MATH 732: CUBIC HYPERSURFACES

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1. Why cubic hypersurfaces?

Algebraic geometry starts with cubic polynomial equations.
- D. Huybrechts

A cubic hypersurface is a subvariety of projective space defined by the vanishing of a single homogeneous polynomial of degree 3. Degree 1 hypersurfaces are isomorphic to projective space and degree 2 hypersurfaces correspond to quadratic forms, so their geometry is almost entirely studied via linear algebra. Cubic hypersurfaces give the first instances of many phenomena in algebraic geometry.

For example, cubic curves form positive dimensional moduli spaces. In other words, their geometry varies continuously (controlled by the famous *j-invariant*). In general, moduli spaces of cubic hypersurfaces give insight into the challenges and the tools needed to construct many other moduli spaces. For example, they are a great examples of moduli of curves, GIT moduli spaces, and K-stability.

Cubics are also a great place to study problems about subvarieties and cycles. The *27 lines on a cubic surface* is one of the most famous facts in algebraic geometry. Fano varieties of lines are a great stepping off point for learning about Chern classes, Hilbert schemes, deformation theory, enumerative geometry. Many "second courses in algebraic geometry" start with this result. Cubic hypersurfaces are also exquisite locales for studying big conjectures on cycles in algebraic geometry including the Hodge conjecture and the conjectural Bloch-Beilinson filtration. Moreover, in the relatively recent industry of Hyperkähler geometry, some of the most important examples are given by moduli spaces of cycles (and objects) on cubic fourfolds.

The rationality problem for cubics has been a central topic in algebraic geometry for the last 50 years. In terms of motivation for doing mathematics, this problem has many of the greatest features: it is easy to express the problem, there have been previous breakthroughs that have had broadly significance in algebraic geometry, there are interesting conjectures that are both beautiful and challenging, there have been exciting but failed recent attempts, and it is not obvious if it will be a problem that is solved in the next 10 or 100 years.